On a Biharmonic Equation Involving Nearly Critical Exponent *

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Abstract. This paper is concerned with a biharmonic equation under the Navier boundary condition $(P_{\mp\varepsilon})$: $\Delta^2 u = u^{\frac{n+4}{n-4}\mp\varepsilon}$, $u>0$ in Ω and $u=\Delta u=0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n\geq 5$, and $\varepsilon>0$. We study the asymptotic behavior of solutions of $(P_{-\varepsilon})$ which are minimizing for the Sobolev quotient as ε goes to zero. We show that such solutions concentrate around a point $x_0\in\Omega$ as $\varepsilon\to 0$, moreover x_0 is a critical point of the Robin's function. Conversely, we show that for any nondegenerate critical point x_0 of the Robin's function, there exist solutions of $(P_{-\varepsilon})$ concentrating around x_0 as $\varepsilon\to 0$. Finally we prove that, in contrast with what happened in the subcritical equation $(P_{-\varepsilon})$, the supercritical problem $(P_{+\varepsilon})$ has no solutions which concentrate around a point of Ω as $\varepsilon\to 0$.

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1 Introduction and Results

In this paper, we are concerned with the following semilinear biharmonic equation under the Navier boundary condition

$$(P_{\mp\varepsilon})\quad \begin{cases} \Delta^2 u = u^{p\mp\varepsilon},\ u>0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n\geq 5$, ε is a small positive parameter, and $p+1 = 2n/(n-4)$ is the critical Sobolev exponent of the embedding $H^2(\Omega)\cap H^1_0(\Omega)\hookrightarrow L^{p+1}(\Omega)$. When the biharmonic operator in $(P_{\mp\varepsilon})$ is replaced by the Laplacian operator, there are many works devoted to the study of the contrepart of $(P_{\mp\varepsilon})$, see for example [1], [2], [5], [7], [11], [13], [14], [17], [18], [20], [21], [23] and the references therein.

When $\varepsilon\in(0,p)$, the mountain pass lemma proves the existence of solution to $(P_{-\varepsilon})$ for any domain Ω. When $\varepsilon=0$, the situation is more complex, Van Der Vorst showed in [24] that if Ω is starshaped (P_0) has no solution whereas Ebobisse and Ould Ahmedou proved in [15] that (P_0) has a solution provided that some homology group of Ω is nontrivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a solution exists

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show [16].
In view of this qualitative change in the situation when $\varepsilon = 0$, it is interesting to study the asymptotic behavior of the subcritical solution u_ε of $(P_{-\varepsilon})$ as $\varepsilon \to 0$. Chou and Geng [12] made the first study, when Ω is a convex domain. The aim of the first result of this paper is to remove the convexity assumption on Ω. To state this result, we need to introduce some notation.
Let us define on Ω the following Robin's function

$$\varphi(x) = H(x,x), \quad \text{with} \quad H(x,y) = \frac{1}{|x-y|^{n-4}} - G(x,y), \text{ for } (x,y) \in \Omega \times \Omega,$$

where G is the Green's function of Δ^2, that is,

$$\forall x \in \Omega \quad \begin{cases} \Delta^2 G(x,.) = c_n \delta_x & \text{in } \Omega \\ \Delta G(x,.) = G(x,.) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_x denotes the Dirac mass at x and $c_n = (n-4)(n-2)|S^{n-1}|$.
Let

$$\delta_{a,\lambda}(x) = \frac{c_0 \lambda^{\frac{n-4}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-4}{2}}}, \; c_0 = [(n-4)(n-2)n(n+2)]^{(n-4)/8}, \; \lambda > 0, \; a \in \mathbb{R}^n \tag{1.1}$$

It is well known (see [19]) that $\delta_{a,\lambda}$ are the only solutions of

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0 \text{ in } \mathbb{R}^n, \quad \text{with } u \in L^{p+1}(\mathbb{R}^n) \quad \text{and } \Delta u \in L^2(\mathbb{R}^n)$$

and are also the only minimizers of the Sobolev inequality on the whole space, that is

$$S = \inf\{|\Delta u|^2_{L^2(\mathbb{R}^n)} |u|^{-2}_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}, \; s.t. \, \Delta u \in L^2, u \in L^{\frac{2n}{n-4}}, u \neq 0\}. \tag{1.2}$$

We denote by $P\delta_{a,\lambda}$ the projection of the $\delta_{a,\lambda}$'s on $H^2(\Omega) \cap H^1_0(\Omega)$, defined by

$$\Delta^2 P\delta_{a,\lambda} = \Delta^2 \delta_{a,\lambda} \text{ in } \Omega \text{ and } \Delta P\delta_{a,\lambda} = P\delta_{a,\lambda} = 0 \text{ on } \partial\Omega.$$

Let

$$||u|| = \left(\int_\Omega |\Delta u|^2\right)^{1/2}, \qquad u \in H^2(\Omega) \cap H^1_0(\Omega) \tag{1.3}$$

$$(u,v) = \int_\Omega \Delta u \Delta v, \qquad u,v \in H^2(\Omega) \cap H^1_0(\Omega) \tag{1.4}$$

$$|u|_q = |u|_{L^q(\Omega)}. \tag{1.5}$$

Now we state the first result of this paper.

Theorem 1.1 *Assume that $n \geq 6$. Let (u_ε) be a solution of $(P_{-\varepsilon})$, and assume that*

$$(H) \qquad ||u_\varepsilon||^2 |u_\varepsilon|^{-2}_{p+1-\varepsilon} \to S \text{ as } \varepsilon \to 0,$$

where S is the best Sobolev constant in $\mathbb{R}^n$ defined by (1.2)*. Then (up to a subsequence) there exist $a_\varepsilon \in \Omega$, $\lambda_\varepsilon > 0$, $\alpha_\varepsilon > 0$ and v_ε such that u_ε can be written as*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$$

with $\alpha_\varepsilon \to 1$, $||v_\varepsilon|| \to 0$, $a_\varepsilon \in \Omega$ and $\lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to +\infty$ as $\varepsilon \to 0$.
In addition, a_ε converges to a critical point $x_0 \in \Omega$ of φ and we have

$$\lim_{\varepsilon\to 0} \varepsilon ||u_\varepsilon||^2_{L^\infty(\Omega)} = (c_1 c_0^2/c_2)\varphi(x_0),$$

where $c_1 = c_0^{2n/(n-4)} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+4)/2}}$, $c_2 = (n-4)c_0^{2n/(n-4)} \int_{\mathbb{R}^n} \frac{log(1+|x|^2)(1-|x|^2)}{(1+|x|^2)^{n+1}}dx$ and c_0 is defined in (1.1)*.*

Remark 1.2 *It is important to point out that in the Laplacian case (see [17]), the method of moving planes has been used to show that blowup points are away from the boundary of domain. The process is standard if domains are convex. For nonconvex regions, the method of moving planes still works in the Laplacian case through the applications of Kelvin transformations [17]. For $(P_{-\varepsilon})$, the method of moving planes also works for convex domains [12]. However, for nonconvex domains, a Kelvin transformation does not work for $(P_{-\varepsilon})$ because the Navier boundary condition is not invariant under the Kelvin transformation of biharmonic operator. Our method here is essential in overcoming the difficulty arising from the nonhomogeneity of Navier boundary condition under the Kelvin transformation.*

Our next result provides a kind of converse to Theorem 1.1.

Theorem 1.3 *Assume that $n \geq 6$, and $x_0 \in \Omega$ is a nondegenerate critical point of φ. Then there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, $(P_{-\varepsilon})$ has a solution of the form*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$$

with $\alpha_\varepsilon \to 1$, $||v_\varepsilon|| \to 0$, $a_\varepsilon \to x_0$ and $\lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to +\infty$ as $\varepsilon \to 0$.

In view of the above results, a natural question arises: are equivalent results true for slightly supercritical exponent?
The aim of the next result is to answer this question.

Theorem 1.4 *Let Ω be any smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$. Then $(P_{+\varepsilon})$ has no solution u_ε of the form*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$$

with $||v_\varepsilon|| \to 0$, $\alpha_\varepsilon \to 1$, $a_\varepsilon \in \Omega$ and $\lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to +\infty$ as $\varepsilon \to 0$.

The proofs of our results are based on the same framework and methods of [22], [23] and [7]. The next section will be devoted to prove Theorem 1.1, while Theorems 1.3 and 1.4 are proved in sections 3 and 4 respectively.

2 Proof of Theorem 1.1

Before starting the proof of Theorem 1.1, we need some preliminary results

Proposition 2.1 *[10] Let $a \in \Omega$ and $\lambda > 0$ such that $\lambda d(a, \partial\Omega)$ is large enough. For $\theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$, we have the following estimates*

$$(a) \quad 0 \leq \theta_{(a,\lambda)} \leq \delta_{(a,\lambda)}, \qquad (b) \quad \theta_{(a,\lambda)} = c_0 \frac{H(a,.)}{\lambda^{\frac{n-4}{2}}} + f_{(a,\lambda)},$$

where $f_{(a,\lambda)}$ satisfies

$$f_{(a,\lambda)} = O\left(\frac{1}{\lambda^{\frac{n}{2}} d^{n-2}}\right), \quad \lambda \frac{\partial f_{(a,\lambda)}}{\partial \lambda} = O\left(\frac{1}{\lambda^{\frac{n}{2}} d^{n-2}}\right), \quad \frac{1}{\lambda} \frac{\partial f_{(a,\lambda)}}{\partial a} = O\left(\frac{1}{\lambda^{\frac{n+2}{2}} d^{n-1}}\right),$$

where d is the distance $d(a, \partial\Omega)$,

$$(c) \qquad | \theta_{(a,\lambda)} |_{L^{\frac{2n}{n-4}}} = O\Big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\Big), \quad | \lambda \frac{\partial \theta_{(a,\lambda)}}{\partial \lambda} |_{L^{\frac{2n}{n-4}}} = O\Big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\Big),$$

$$|| \theta_{(a,\lambda)} || = O\Big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\Big), \quad | \frac{1}{\lambda} \frac{\partial \theta_{(a,\lambda)}}{\partial a} |_{L^{\frac{2n}{n-4}}} = O\Big(\frac{1}{(\lambda d)^{\frac{n-2}{2}}}\Big).$$

Proposition 2.2 *Let u_ε be a solution of $(P_{-\varepsilon})$ which satisfies (H). Then, we have*

$$(a) \quad ||u_\varepsilon||^2 \to S^{n/4}, \quad (b) \quad \int u_\varepsilon^{p+1-\varepsilon} \to S^{n/4}.$$

Proof. Since u_ε is a solution of $(P_{-\varepsilon})$, then we have $||u_\varepsilon||^2 = \int u_\varepsilon^{p+1-\varepsilon}$. Thus, using the assumption (H), we derive that

$$||u_\varepsilon||^2 |u_\varepsilon|_{p+1-\varepsilon}^{-2} = ||u_\varepsilon||^{\frac{2(p-1-\varepsilon)}{p+1-\varepsilon}} = S + o(1).$$

Therefore $||u_\varepsilon||^2 = \int u_\varepsilon^{p+1-\varepsilon} = S^{n/4} + o(1)$. The result follows. □

Proposition 2.3 *Let u_ε be a solution of $(P_{-\varepsilon})$ which satisfies (H), and let $x_\varepsilon \in \Omega$ such that $u_\varepsilon(x_\varepsilon) = |u_\varepsilon|_{L^\infty} := M_\varepsilon$. Then, for ε small, we have*
(a) $M_\varepsilon^\varepsilon = 1 + o(1)$.
(b) *u_ε can be written as*

$$u_\varepsilon = P\delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon} + \tilde{v}_\varepsilon,$$

with $||\tilde{v}_\varepsilon|| \to 0$, where $\tilde{\lambda}_\varepsilon = c_0^{2/(4-n)} M_\varepsilon^{(p-1-\varepsilon)/4}$.

Proof. First of all, we prove that $M_\varepsilon \to +\infty$ as $\varepsilon \to 0$. To this end, arguing by contradiction, we suppose that M_{ε_n} remains bounded for a sequence $\varepsilon_n \to 0$ as $n \to +\infty$. Then, in view of elliptic regularity theory, we can extract a subsequence, still denoted by u_{ε_n}, which converges uniformly to a limit u_o. By Proposition 2.2, $u_0 \neq 0$, hence by taking limit in (H) we find that

u_0 achieves a best Sobolev constant S, a contradiction to the fact that S is never achieved on a bounded domain [25].
Now we define the rescaled functions

$$\omega_\varepsilon(y) = M_\varepsilon^{-1} u_\varepsilon \left(x_\varepsilon + M_\varepsilon^{(1+\varepsilon-p)/4} y \right), \quad y \in \Omega_\varepsilon = M_\varepsilon^{(p-1-\varepsilon)/4}(\Omega - x_\varepsilon), \tag{2.1}$$

ω_ε satisfies

$$\left\{ \begin{array}{lcccc} \Delta^2 \omega_\varepsilon & = & \omega_\varepsilon^{p-\varepsilon}, & 0 < \omega_\varepsilon \leq 1 & \text{in} \quad \Omega_\varepsilon \\ \omega_\varepsilon(0) & = & 1, & \Delta\omega_\varepsilon = \omega_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{array} \right. \tag{2.2}$$

Following the same argument as in Lemma 2.3 [8], we have

$$M_\varepsilon^{(p-1-\varepsilon)/4} d(x_\varepsilon, \partial\Omega) \to +\infty \quad \text{as } \varepsilon \to 0.$$

Then it follows from standard elliptic theory that there exists a positive function ω such that (after passing to a subsequence) $\omega_\varepsilon \to \omega$ in $C^4_{loc}(\mathbb{R}^n)$, and ω satisfies

$$\left\{ \begin{array}{lccc} \Delta^2 \omega & = & \omega^p, & 0 \leq \omega \leq 1 \quad \text{in } \mathbb{R}^n \\ \omega(0) & = & 1, & \nabla\omega(0) = 0. \end{array} \right.$$

It follows from [19] that ω writes as

$$\omega(y) = \delta_{0,\alpha_n}(y), \quad \text{with } \alpha_n = c_0^{2/(4-n)}.$$

Observe that, for $y = M_\varepsilon^{(p-1-\varepsilon)/4}(x - x_\varepsilon)$, we have

$$M_\varepsilon \delta_{0,\alpha_n}(y) = \frac{M_\varepsilon c_0 \alpha_n^{(n-4)/2}}{\left(1 + \alpha_n^2 M_\varepsilon^{(p-1-\varepsilon)/2} |x - x_\varepsilon|^2\right)^{(n-4)/2}} = M_\varepsilon^{\varepsilon(n-4)/8} \delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon} \tag{2.3}$$

with $\tilde{\lambda}_\varepsilon = \alpha_n M_\varepsilon^{(p-1-\varepsilon)/4}$. Then,

$$w_\varepsilon(y) - \delta_{0,\alpha_n}(y) = M_\varepsilon^{-1} \left(u_\varepsilon(x) - M_\varepsilon^{\varepsilon(n-4)/8} \delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon} \right).$$

Let us define

$$u_\varepsilon^1(x) = u_\varepsilon(x) - M_\varepsilon^{\varepsilon(n-4)/8} P\delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}(x),$$

we need to compute

$$||u_\varepsilon^1||^2 = ||u_\varepsilon||^2 + M_\varepsilon^{\varepsilon(n-4)/4} ||P\delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}||^2 - 2M_\varepsilon^{\varepsilon(n-4)/8} (u_\varepsilon, P\delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}).$$

On one hand, we have

$$\begin{aligned} (u_\varepsilon, P\delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}) &= \int_\Omega u_\varepsilon(x) \delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}^{(n+4)/(n-4)}(x) \\ &= \int_{\Omega_\varepsilon} u_\varepsilon(x_\varepsilon + M_\varepsilon^{(1+\varepsilon-p)/4} y) \delta_{x_\varepsilon, \tilde{\lambda}_\varepsilon}^{(n+4)/(n-4)}(x_\varepsilon + M_\varepsilon^{(1+\varepsilon-p)/4} y) M_\varepsilon^{n(1+\varepsilon-p)/4} dy \\ &= \int_{\Omega_\varepsilon} M_\varepsilon^{\varepsilon(n-4)/8} w_\varepsilon(y) \delta_{0,\alpha_n}^{(n+4)/(n-4)}(y) dy \\ &= \int_{B(0,R)} M_\varepsilon^{\varepsilon(n-4)/8} w_\varepsilon(y) \delta_{0,\alpha_n}^{(n+4)/(n-4)}(y) dy + \int_{\Omega_\varepsilon \setminus B(0,R)} M_\varepsilon^{\varepsilon(n-4)/8} w_\varepsilon(y) \delta_{0,\alpha_n}^{(n+4)/(n-4)}(y) dy, \end{aligned}$$

where R is a large positive constant such that $\int_{\mathbb{R}^n\setminus B(0,R)}\delta_{0,\alpha_n}^{2n/(n-4)}=o(1)$.
Since

$$\int_{\Omega_\varepsilon} w_\varepsilon^{2n/(n-4)} = M_\varepsilon^{-\varepsilon n/4}\int_\Omega u_\varepsilon^{2n/(n-4)} \leq c,$$

using Holder's inequality we derive that

$$\int_{\Omega_\varepsilon\setminus B(0,R)} w_\varepsilon \delta_{0,\alpha_n}^{(n+4)/(n-4)} = o(1).$$

Now, since $w_\varepsilon \to \delta_{0,\alpha_n}$ in $C^4_{loc}(\mathbb{R}^n)$, we obtain

$$(u_\varepsilon, P\delta_{x_\varepsilon,\tilde\lambda_\varepsilon}) = M_\varepsilon^{\varepsilon(n-4)/8}(S^{n/4}+o(1)).$$

On the other hand, one can easy verify that

$$||P\delta_{x_\varepsilon,\tilde\lambda_\varepsilon})||^2 = S^{n/4}+o(1).$$

Thus

$$||u_\varepsilon^1||^2 = ||u_\varepsilon||^2 - M_\varepsilon^{\varepsilon(n-4)/4}(S^{n/4}+o(1)), \tag{2.4}$$

and, using the fact that $||u_\varepsilon^1||^2\geq 0$ and Proposition 2.2, we derive that

$$M_\varepsilon^{\varepsilon(n-4)/4} \leq 1+o(1).$$

But, since $M_\varepsilon\to\infty$, we have $M_\varepsilon^\varepsilon \geq 1$ and therefore claim (a) follows.
Now we are going to prove claim (b). Observe that, using Proposition 2.2 and claim (a), (2.4) becomes

$$||u_\varepsilon^1||^2 = (S^{n/4}+o(1)) - (S^{n/4}+o(1)) = o(1).$$

Thus claim (b) follows. □

Proposition 2.4 *Let u_ε be a solution of $(P_{-\varepsilon})$ which satisfies (H). Then, there exist $a_\varepsilon\in\Omega$, $\alpha_\varepsilon>0$, $\lambda_\varepsilon>0$ and v_ε such that*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$$

with $\alpha_\varepsilon\to 1$, $|a_\varepsilon - x_\varepsilon|\to 0$, $\lambda_\varepsilon d(a_\varepsilon,\partial\Omega)\to\infty$, $\tilde\lambda_\varepsilon/\lambda_\varepsilon\to 1$ and $||v_\varepsilon||\to 0$. Furthermore, v_ε satisfies

$$(V_0) \qquad (v, P\delta_{a_\varepsilon,\lambda_\varepsilon}) = (v, \partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial\lambda_\varepsilon) = 0,\ (v, \partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial a) = 0.$$

Proof. By Proposition 2.3, u_ε can be written as $u_\varepsilon = P\delta_{x_\varepsilon,\tilde\lambda_\varepsilon} + \tilde v_\varepsilon$ with $||\tilde v_\varepsilon||\to 0$, $\tilde\lambda_\varepsilon d(x_\varepsilon,\partial\Omega)\to\infty$ as $\varepsilon\to 0$. Thus, the following minimization problem

$$\min\{||u_\varepsilon - \alpha P\delta_{a,\lambda}||, \alpha>0, a\in\Omega, \lambda>0\}$$

has a unique solution $(\alpha_\varepsilon, a_\varepsilon, \lambda_\varepsilon)$. Then, for $v_\varepsilon = u_\varepsilon - \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon}$, we have v_ε satisfies (V_0). From the two forms of u_ε, one can easy verify that

$$||P\delta_{x_\varepsilon,\tilde\lambda_\varepsilon} - P\delta_{a_\varepsilon,\lambda_\varepsilon}|| = o(1).$$

Therefore, we derive that $|a_\varepsilon - x_\varepsilon| = o(1)$ and $\tilde\lambda_\varepsilon/\lambda_\varepsilon = 1+o(1))$. The result follows. □

Next, we state a result which its proof is similar to the proof of Lemma 2.3 of [7], so we will omit it.

Lemma 2.5 *$\lambda_\varepsilon^\varepsilon = 1 + o(1)$ as ε goes to zero implies that*

$$\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2}} = O\left(\varepsilon Log(1 + \lambda_\varepsilon^2 |x - a_\varepsilon|^2)\right) \quad in \quad \Omega.$$

We are now able to study the v_ε-part of u_ε solution of $(P_{-\varepsilon})$.

Proposition 2.6 *Let (u_ε) be a solution of $(P_{-\varepsilon})$ which satisfies (H). Then v_ε occuring in Proposition 2.4 satisfies*

$$||v_\varepsilon|| \leq C\varepsilon + C\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\,(if\ n < 12) + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2} - \varepsilon(n-4)}}\,(if\ n \geq 12)\right),$$

where C is a positive constant independent of ε.

Proof. Multiplying $(P_{-\varepsilon})$ by v_ε and integrating on Ω, we obtain

$$\int_\Omega \Delta u_\varepsilon . \Delta v_\varepsilon - \int_\Omega u_\varepsilon^{p-\varepsilon} v_\varepsilon = 0.$$

Thus

$$\int_\Omega |\Delta v_\varepsilon|^2 - \int_\Omega \left((\alpha_\varepsilon P\delta_\varepsilon)^{p-\varepsilon} + (p-\varepsilon)(\alpha_\varepsilon P\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon + O\left(\delta_\varepsilon^{p-2-\varepsilon} v_\varepsilon^2 \chi_{|v_\varepsilon| < \delta_\varepsilon} + |v_\varepsilon|^{p-\varepsilon}\right)\right) v_\varepsilon = 0.$$

Using Lemma 2.5, we find

$$Q_\varepsilon(v_\varepsilon, v_\varepsilon) - f_\varepsilon(v_\varepsilon) + o(||v_\varepsilon||^2) = 0, \tag{2.5}$$

with

$$Q_\varepsilon(v, v) = ||v||^2 - (p-\varepsilon) \int (\alpha_\varepsilon P\delta_\varepsilon)^{p-1-\varepsilon} v^2$$

and

$$f_\varepsilon(v) = \int (\alpha_\varepsilon P\delta_\varepsilon)^{p-\varepsilon} v.$$

We observe that

$$\begin{aligned}
Q_\varepsilon(v, v) &= ||v||^2 - p\int_\Omega (\alpha_\varepsilon P\delta_\varepsilon)^{p-1-\varepsilon} v^2 + O\left(\varepsilon ||v||^2\right) \\
&= ||v||^2 - p\alpha_\varepsilon^{p-1-\varepsilon} \int_\Omega \left(\delta_\varepsilon^{p-1-\varepsilon} + O\left(\delta_\varepsilon^{p-2-\varepsilon}\theta_\varepsilon\right)\right) v^2 + o\left(||v||^2\right) \\
&= ||v||^2 - \frac{p\alpha_\varepsilon^{p-1-\varepsilon}}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2}} \int_\Omega \delta_\varepsilon^{p-1} v^2 + O\left(\int_\Omega \left|\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2}}\right| \delta_\varepsilon^{p-1} |v|^2\right) + o\left(||v||^2\right)
\end{aligned}$$

Using Lemma 2.5 and the fact that $\alpha_\varepsilon \to 1$, we find

$$Q_\varepsilon(v, v) = Q_0(v, v) + o\left(||v||^2\right),$$

with

$$Q_0(v,v) = ||v||^2 - p\int_\Omega \delta_\varepsilon^{p-1} v^2.$$

According to [6], Q_0 is coercive, that is, there exists some constant $c > 0$ independent of ε, for ε small enough, such that

$$Q_0(v,v) \geq c||v||^2 \quad \forall v \in E_{(a_\varepsilon,\lambda_\varepsilon)}, \tag{2.6}$$

where

$$E_{(a_\varepsilon,\lambda_\varepsilon)} = \{v \in E/v \text{ satisfies } (V_0)\}, \tag{2.7}$$

(V_0) is the condition defined in Proposition 2.4.
We also observe that

$$\begin{aligned} f_\varepsilon(v) &= \alpha_\varepsilon^{p-\varepsilon}\int_\Omega \left(\delta_\varepsilon^{p-\varepsilon} + O\left(\delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon\right)\right) v \\ &= \alpha_\varepsilon^{p-\varepsilon}\left[\frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2}}\int_\Omega \delta_\varepsilon^p v + O\left(\varepsilon\int_\Omega Log(1+\lambda_\varepsilon^2|x-a_\varepsilon|^2)\delta_\varepsilon^p|v| + \int_\Omega \delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon|v|\right)\right]. \end{aligned}$$

The last equality follows from Lemma 2.5. Therefore we can write, with $B = B(a_\varepsilon, d_\varepsilon)$

$$\begin{aligned} f_\varepsilon(v) &\leq c\left(\varepsilon||v|| + \int_B \delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon|v| + \int_{\mathbb{R}^n\setminus B}\delta_\varepsilon^p|v|\right) \\ &\leq c||v||\left(\varepsilon + |\theta_\varepsilon|_{L^\infty}\left(\int_B \delta_\varepsilon^{(p-1-\varepsilon)\frac{2n}{n+4}}\right)^{\frac{n+4}{2n}} + \left(\int_{\mathbb{R}^n\setminus B}\delta_\varepsilon^{\frac{2n}{n-4}}\right)^{\frac{n+4}{2n}}\right). \end{aligned}$$

We notice that

$$\int_{\mathbb{R}^n\setminus B}\delta_\varepsilon^{2n/(n-4)} = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^n}\right) \tag{2.8}$$

and

$$|\theta_\varepsilon|_{L^\infty}\left(\int_B \delta_\varepsilon^{\frac{2n(p-1-\varepsilon)}{n+4}}\right)^{\frac{n+4}{2n}} \leq \frac{c}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}-\varepsilon(n-4)}}(\text{ if } n \geq 12) + \frac{c}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}(\text{ if } n < 12). \tag{2.9}$$

Thus we obtain

$$|f_\varepsilon(v)| \leq C||v||\left(\varepsilon + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}(\text{ if } n < 12) + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}-\varepsilon(n-4)}}(\text{ if } n \geq 12)\right) \tag{2.10}$$

Combining (2.5), (2.6) and (2.10), we obtain the desired estimate. □

Next we prove the following crucial result :

Proposition 2.7 *For $u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ solution of $(P_{-\varepsilon})$ with $\lambda_\varepsilon^\varepsilon = 1 + o(1)$ as ε goes to zero, we have the following estimate*

$$(a) \qquad c_2\varepsilon + O(\varepsilon^2) - c_1\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}} + o\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right) = 0.$$

and for $n \geq 6$, we also have

$$(b) \qquad \frac{c_3}{\lambda_\varepsilon^{n-3}}\frac{\partial H}{\partial a_\varepsilon}(a_\varepsilon,a_\varepsilon) + O(\varepsilon^2) + o\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-3}}\right) = 0,$$

where c_1, c_2 are the constants defined in Theorem 1.1, and where c_3 is a positive constant.

Proof. We start by giving the proof of Claim (a). Multiplying the equation $(P_{-\varepsilon})$ by $\lambda_\varepsilon(\partial P\delta_\varepsilon)/(\partial\lambda_\varepsilon)$ and integrating on Ω, we obtain

$$\begin{aligned} 0 &= \int_\Omega \Delta^2 u_\varepsilon \lambda_\varepsilon \frac{\partial P\delta_\varepsilon}{\partial\lambda} - \int_\Omega u_\varepsilon^{p-\varepsilon}\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} \\ &= \alpha_\varepsilon\int_\Omega \delta_\varepsilon^p\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} - \int_\Omega \Big[(\alpha_\varepsilon P\delta_\varepsilon)^{p-\varepsilon} + (p-\varepsilon)(\alpha_\varepsilon P\delta_\varepsilon)^{p-1-\varepsilon}v_\varepsilon \\ &\quad + O\left(\delta_\varepsilon^{p-2-\varepsilon}|v_\varepsilon|^2 + |v_\varepsilon|^{p-\varepsilon}\chi_{\delta_\varepsilon\leq|v_\varepsilon|}\right)\Big]\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda}. \end{aligned} \tag{2.11}$$

We estimate each term of the right-hand side in (2.11). First, using Proposition 2.1, we have

$$\begin{aligned} &\int_{B^c}\delta_\varepsilon^p\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda_\varepsilon} \leq c\int_{B^c}\delta_\varepsilon^{p+1} = O(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^n}) \\ &\int_B \delta_\varepsilon^p\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} = \int_B\delta_\varepsilon^p\lambda_\varepsilon\frac{\partial\delta_\varepsilon}{\partial\lambda} + \frac{(n-4)c_0}{2\lambda_\varepsilon^{(n-4)/2}}\int_B\delta_\varepsilon^p H - \int_B\delta_\varepsilon^p\lambda_\varepsilon\frac{\partial f_\varepsilon}{\partial\lambda}, \end{aligned}$$

with $B = B(a_\varepsilon, d_\varepsilon)$. Expanding $H(a_\varepsilon, .)$ around a_ε and using Proposition 2.1, we obtain

$$\int_B\delta_\varepsilon^p\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^n}\right) + \frac{(n-4)c_0}{2\lambda_\varepsilon^{(n-4)/2}}H(a_\varepsilon,a_\varepsilon)\int_B\delta_\varepsilon^p + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right).$$

Therefore, estimating the integral, we obtain

$$\int_\Omega\delta_\varepsilon^p\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} = \frac{n-4}{2}c_1\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right) \tag{2.12}$$

with $c_1 = c_0^{2n/(n-4)}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+4)/2}}$.
Secondly, we compute

$$\begin{aligned} \int_\Omega(P\delta_\varepsilon)^{p-\varepsilon}\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} &= \int_\Omega\Big[\delta_\varepsilon^{p-\varepsilon} - (p-\varepsilon)\delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon + O\left(\theta_\varepsilon^2\delta_\varepsilon^{p-2-\varepsilon} + \theta_\varepsilon^{p-\varepsilon}\right)\Big]\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial\lambda} \\ &= \int_B\delta_\varepsilon^{p-\varepsilon}\lambda_\varepsilon\frac{\partial\delta_\varepsilon}{\partial\lambda} - \int_B\delta_\varepsilon^{p-\varepsilon}\lambda_\varepsilon\frac{\partial\theta_\varepsilon}{\partial\lambda} - (p-\varepsilon)\int_B\delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon\lambda\frac{\partial\delta_\varepsilon}{\partial\lambda} \\ &\quad + O\left(\int_\Omega\delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon|\lambda_\varepsilon\frac{\partial\theta_\varepsilon}{\partial\lambda}| + \int_\Omega\theta_\varepsilon^2\delta_\varepsilon^{p-1-\varepsilon} + \int_\Omega\theta_\varepsilon^{p-\varepsilon}\delta_\varepsilon + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-\varepsilon\frac{n-4}{2}}}\right) \end{aligned} \tag{2.13}$$

and we have to estimate each term of the right hand-side of (2.13).
Using the fact that $\lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial \lambda} = \frac{n-4}{2}\left(\frac{1-\lambda_\varepsilon^2|x-a_\varepsilon|^2}{1+\lambda_\varepsilon^2|x-a_\varepsilon|^2}\right)\delta_\varepsilon$, we derive that

$$\begin{aligned}\int_B \delta_\varepsilon^{p-\varepsilon}\lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial \lambda} &= \frac{n-4}{2}\frac{c_0^{p+1-\varepsilon}}{\lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}}\int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^{n-\frac{\varepsilon(n-4)}{2}}}\frac{1-|x|^2}{1+|x|^2}dx + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-\varepsilon(n-4)}}\right)\\ &= \frac{n-4}{2\lambda_\varepsilon^{\varepsilon(n-4)/2}}\left(-c_2\varepsilon + O(\varepsilon^2)\right) + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-\varepsilon(n-4)}}\right)\end{aligned} \tag{2.14}$$

with $c_2 = \frac{n-4}{2}c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{Log(1+|x|^2)}{(1+|x|^2)^n}\frac{|x|^2-1}{|x|^2+1}dx > 0$.
For the other terms in (2.13), using Proposition 2.1, we have

$$\begin{aligned}\int_B \delta_\varepsilon^{p-\varepsilon}\lambda_\varepsilon \frac{\partial \theta_\varepsilon}{\partial \lambda} &= \int_B \delta_\varepsilon^{p-\varepsilon}\lambda_\varepsilon \frac{\partial}{\partial \lambda}\left(\frac{c_0 H}{\lambda_\varepsilon^{(n-4)/2}} - f_\varepsilon\right)\\ &= -\frac{n-4}{2}c_0^{p+1-\varepsilon}\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{(n-4)/2}}\int_B\left(\frac{\lambda_\varepsilon}{1+\lambda_\varepsilon^2|x-a_\varepsilon|^2}\right)^{(p-\varepsilon)\frac{(n-4)}{2}} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right)\\ &= -\frac{n-4}{2}\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}}\frac{1}{\lambda_\varepsilon^{\varepsilon(n-4)/2}}\int_{B(0,\lambda d)}\frac{c_0^{p+1-\varepsilon}}{(1+|x|^2)^{\frac{n+4}{2}-\varepsilon\frac{n-4}{2}}} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right)\\ &= -\frac{n-4}{2}c_1\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}}\frac{1}{\lambda_\varepsilon^{\varepsilon(n-4)/2}} + O\left(\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^{n-4}} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right)\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}(p-\varepsilon)\int_B \delta_\varepsilon^{p-1-\varepsilon}\theta_\varepsilon\lambda_\varepsilon\frac{\partial \delta_\varepsilon}{\partial \lambda} &= (p-\varepsilon)\int_B \delta_\varepsilon^{p-1-\varepsilon}\frac{c_0}{\lambda_\varepsilon^{(n-4)/2}}H\lambda_\varepsilon\frac{\partial \delta_\varepsilon}{\partial \lambda} + O\left(\int_B \delta_\varepsilon^{p-\varepsilon}f_\varepsilon\right)\\ &= (p-\varepsilon)\frac{c_0}{\lambda_\varepsilon^{(n-4)/2}}H(a_\varepsilon,a_\varepsilon)\int_B \delta_\varepsilon^{p-1-\varepsilon}\lambda_\varepsilon\frac{\partial \delta_\varepsilon}{\partial \lambda} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right)\\ &= \frac{c_0(p-\varepsilon)}{\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}}\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}}\int_{B(0,\lambda_\varepsilon d_\varepsilon)}\frac{(n-4)(1-|x|^2)}{2(1+|x|^2)^{\frac{n+6}{2}-\varepsilon\frac{n-4}{2}}} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right)\\ &= -\frac{n-4}{2}\frac{c_1}{\lambda_\varepsilon^{\varepsilon(n-4)/2}}\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}} + O\left(\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^{n-4}} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}}\right).\end{aligned}$$

(2.13), (2.14), (2.15) and additional integral estimates of the same type provide us with the expansion

$$\begin{aligned}\int_\Omega (P\delta_\varepsilon)^{p-\varepsilon}\lambda_\varepsilon\frac{\partial P\delta_\varepsilon}{\partial \lambda} =& \frac{n-4}{2\lambda_\varepsilon^{\varepsilon(n-4)/2}}\Bigg[-c_2\varepsilon + O(\varepsilon^2) + 2c_1\frac{H(a_\varepsilon,a_\varepsilon)}{\lambda_\varepsilon^{n-4}}\\ &+O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-2}} + (if\, n=5)\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right)\Bigg].\end{aligned} \tag{2.16}$$

We note that

$$\begin{aligned}\int_\Omega (P\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial P\delta_\varepsilon}{\partial\lambda} &= \int_\Omega \left(\delta_\varepsilon^{p-1-\varepsilon} + O\left(\delta_\varepsilon^{p-2-\varepsilon}\theta_\varepsilon\right)\right) v_\varepsilon \lambda_\varepsilon \frac{\partial P\delta_\varepsilon}{\partial\lambda}\\
&= \int_\Omega (\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda} - \int_\Omega (\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial \theta_\varepsilon}{\partial\lambda} + O\left(\int_\Omega \delta_\varepsilon^{p-1-\varepsilon}|v_\varepsilon|\theta_\varepsilon\right)\\
&= \int_\Omega (\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda} + O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon^2)^{(n-4)/2}}\left(\int_B \delta_\varepsilon^{(p-1-\varepsilon)\frac{2n}{n+4}}\right)^{(n+4)/2n}\right)\\
&\quad + O\left(||v_\varepsilon||\left(\int_{B^c} \delta_\varepsilon^{(p-\varepsilon)\frac{2n}{n+4}}\right)^{(n+4)/(2n)}\right). \qquad (2.17)\end{aligned}$$

Using (2.8) and (2.9), we derive that

$$\begin{aligned}\int_\Omega (P\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial P\delta_\varepsilon}{\partial\lambda} &= \int_\Omega \delta_\varepsilon^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda} + O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}-\varepsilon(n-4)}}\right) \qquad (2.18)\\
&\quad + (\text{if } n<12) O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right).\end{aligned}$$

We also have, using Lemma 2.5

$$\begin{aligned}\int_\Omega (\delta_\varepsilon)^{p-1-\varepsilon} v_\varepsilon \lambda_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda} &= \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\frac{\varepsilon(n-2)}{2}}}\int_\Omega \delta_\varepsilon^{p-1}\lambda_\varepsilon v_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda} + \int_\Omega \left(\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\frac{\varepsilon(n-2)}{2}}}\right)\delta_\varepsilon^{p-1} v_\varepsilon \frac{\partial \delta_\varepsilon}{\partial\lambda}\\
&= O\left(\varepsilon \int_\Omega Log(1+\lambda_\varepsilon^2|x-a_\varepsilon|^2)\delta_\varepsilon^p |v_\varepsilon|\right) = O(\varepsilon||v_\varepsilon||).\end{aligned}$$

Noticing that, in addition, $\lambda_\varepsilon \frac{\partial P\delta_\varepsilon}{\partial\lambda} = O(\delta_\varepsilon)$ and

$$\int_\Omega \delta_\varepsilon^{p-1-\varepsilon}|v_\varepsilon|^2 = O\left(||v_\varepsilon||^2\right), \quad \int_{\delta<|v_\varepsilon|} |v_\varepsilon|^{p-\varepsilon}\delta_\varepsilon = O\left(||v_\varepsilon||^{p+1-\varepsilon}\right). \qquad (2.19)$$

(2.12), (2.16),..., (2.19), Proposition 2.6, Lemma 2.5 and the fact that $\lambda_\varepsilon^\varepsilon = 1 + O(\varepsilon Log\lambda_\varepsilon)$ prove claim (a) of our Proposition.
Now, since the proof of Claim (b) is similar to that of Claim (a), we only point out some necessary changes in the proof. We multiply the equation $(P_{-\varepsilon})$ by $(1/\lambda_\varepsilon)(\partial P\delta_\varepsilon/\partial a)$ and we integrate on Ω, thus we obtain a similar equation as (2.11). As (2.12), we have

$$\begin{aligned}\int_\Omega \delta_\varepsilon^p \frac{1}{\lambda}\frac{\partial P\delta_\varepsilon}{\partial a} &= \int_B \delta_\varepsilon^p \frac{1}{\lambda}\frac{\partial \delta_\varepsilon}{\partial a} - \frac{c_0}{\lambda_\varepsilon^{(n-2)/2}}\int_B \delta_\varepsilon^p \frac{\partial H}{\partial a} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right)\\
&= -\frac{c_1}{2\lambda_\varepsilon^{n-3}}\frac{\partial H}{\partial a}(a_\varepsilon, a_\varepsilon) + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right). \qquad (2.20)\end{aligned}$$

Now, to obtain the similar result as (2.16), we need to estimate the following quantities

$$\int_B \delta_\varepsilon^{p-\varepsilon}\frac{1}{\lambda}\frac{\partial \delta_\varepsilon}{\partial a} = 0$$

$$\int_B \delta_\varepsilon^{p-\varepsilon}\frac{1}{\lambda}\frac{\partial \theta_\varepsilon}{\partial a} = \int_B \delta_\varepsilon^{p-\varepsilon}\left(\frac{c_0}{\lambda_\varepsilon^{(n-2)/2}}\frac{\partial H}{\partial a} - \frac{1}{\lambda}\frac{\partial f}{\partial a}\right)$$
$$= \frac{1}{\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\lambda_\varepsilon^{n-3}}\frac{\partial H}{\partial a}(a_\varepsilon, a_\varepsilon)\left(c_1 + O(\varepsilon)\right) + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right)$$

$$\int \theta\frac{1}{\lambda}\frac{\partial}{\partial a}(\delta_\varepsilon)^{(p-\varepsilon)} = (p-\varepsilon)D\theta(a_\varepsilon)\int \delta_\varepsilon^{p-1-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a}(x - a_\varepsilon) + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right)$$
$$= \frac{1}{\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\lambda_\varepsilon^{n-3}}\frac{\partial H}{\partial y}(a_\varepsilon, a_\varepsilon)\left(c_1 + O(\varepsilon)\right) + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right).$$

These estimates imply that

$$\int_\Omega P\delta_\varepsilon^{p-\varepsilon}\frac{1}{\lambda}\frac{\partial P\delta_\varepsilon}{\partial a} = \frac{-c_1}{\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\lambda_\varepsilon^{n-3}}\frac{\partial H}{\partial a}(a_\varepsilon, a_\varepsilon) + O\left(\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^{n-3}} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-1}}\right) + (if\, n = 6)O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^4}\right). \tag{2.21}$$

Now, it remains to prove the similar result as (2.17). Using the same arguments, we obtain

$$\int_\Omega P\delta_\varepsilon^{p-1-\varepsilon}v_\varepsilon\frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a} = O\left(\varepsilon^2 + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n+4-\varepsilon(n-4)}} + (if\, n < 12)\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{2(n-4)}}\right). \tag{2.22}$$

The proof of claim (b) follows. $\square$

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1 Let (u_ε) be a solution of $(P_{-\varepsilon})$ which satisfies (H). Then, using Proposition 2.4, $u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ with $\alpha_\varepsilon \to 1$, $\lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to \infty$, v_ε satisfies (V_0) and $||v_\varepsilon|| \to 0$. By Propositions 2.3 and 2.4, we have $\lambda_\varepsilon^\varepsilon \to 1$. Now, using claim (a) of Proposition 2.7, we derive that

$$\varepsilon = \frac{c_1}{c_2}\frac{H(a_\varepsilon, a_\varepsilon)}{\lambda_\varepsilon^{n-4}} + o\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right) = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right). \tag{2.23}$$

Now claim (b) implies that

$$\frac{\partial H}{\partial a_\varepsilon}(a_\varepsilon, a_\varepsilon) = o\left(\frac{1}{d_\varepsilon^{n-3}}\right). \tag{2.24}$$

Using (2.24) and the fact that for a near the boundary $\partial H/\partial a(a_\varepsilon, a_\varepsilon) \sim cd^{3-n}(a, \partial\Omega)$, we derive that a_ε is away from the boundary and it converges to a critical point x_0 of φ.
Finally, using (2.23), we obtain

$$\varepsilon\lambda_\varepsilon^{n-4} \to \frac{c_1}{c_2}\varphi(x_0) \text{ as } \varepsilon \to 0.$$

Thus in order to complete the proof of our theorem, it only remains to show that

$$M_\varepsilon := ||u_\varepsilon||_{L^\infty} \sim c_0 \lambda_\varepsilon^{(n-4)/2} \quad \text{as} \quad \varepsilon \to 0. \tag{2.25}$$

Using Propositions 2.3 and 2.4, we derive that $c_0^2 ||u_\varepsilon||_{L^\infty}^{-2} \lambda_\varepsilon^{n-4} \to 1$. Hence (2.25) follows. This concludes the proof of Theorem 1.1. □

3 Proof of Theorem 1.3

Let x_0 be a nondegenerate critical point of φ. It is easy to see that $d(a, \partial\Omega) > d_0 > 0$ for a near x_0. We will take a function $u = \alpha P\delta_{(a,\lambda)} + v$ where $(\alpha - \alpha_0)$ is very small, λ is large enough, $||v||$ is very small, a is close to x_0 and $\alpha_0 = S^{-n/8}$ and we will prove that we can choose the variables (α, λ, a, v) so that u is a critical point of J_ε, where

$$J_\varepsilon(u) = \left(\int_\Omega |\Delta u|^2 \right) \left(\int_\Omega |u|^{p+1-\varepsilon} \right)^{-2/(p+1-\varepsilon)}$$

is the functional corresponding to problem $(P_{-\varepsilon})$.
Let

$$M_\varepsilon = \{(\alpha, \lambda, a, v) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \Omega \times E/ \ |\alpha - \alpha_0| < \nu_0, \ d_a > d_0, \ \lambda > \nu_0^{-1}, \\ \varepsilon Log\lambda < \nu_0, \ ||v|| < \nu_0 \text{ and } v \in E_{(a,\lambda)}\},$$

where ν_0 and d_0 are two suitable positive constants and where $d_a = d(a, \partial\Omega)$.
Let us define the functional

$$K_\varepsilon : M_\varepsilon \to \mathbb{R}, \quad K_\varepsilon(\alpha, a, \lambda, v) = J_\varepsilon(\alpha P\delta_{(a,\lambda)} + v).$$

Notice that (α, λ, a, v) is a critical point of K_ε if and only if $u = \alpha P\delta_{(a,\lambda)} + v$ is a critical point of J_ε on E. So this fact allows us to look for critical points of J_ε by successive optimizations with respect to the different parameters on M_ε.
First, arguing as in Proposition 4 of [22] and using computations performed in the previous sections, we observe that the following problem

$$\min\{J_\varepsilon(\alpha P\delta_{(a,\lambda)} + v), \ v \in E_{(a,\lambda)} \text{ and } ||v|| < \nu_0\}$$

is achieved by a unique function $\overline{v}$ which satisfies the estimate of Proposition 2.6. This implies that there exist A, B and C_i's such that

$$\frac{\partial K_\varepsilon}{\partial v}(\alpha, \lambda, a, \overline{v}) = \nabla J_\varepsilon(\alpha P\delta_{(a,\lambda)} + \overline{v}) = AP\delta_{(a,\lambda)} + B\frac{\partial}{\partial \lambda} P\delta_{(a,\lambda)} + \sum_{i=1}^n C_i \frac{\partial}{\partial a_i} P\delta_{(a,\lambda)}, \tag{3.1}$$

where a_i is the i^{th} component of a.
Now, we need to estimate the constants A, B, and C_i's. For this purpose, we take the scalar

product of $\nabla J_\varepsilon(\alpha P\delta_{(a,\lambda)}+\overline{v})$ with $P\delta_{(a,\lambda)}$, $(\partial P\delta_{(a,\lambda)})/(\partial\lambda)$ and $(\partial P\delta_{(a,\lambda)})/(\partial a_i)$ with $i=1,...,n$. Thus, we get a quasi-diagonal system whose coefficients are given by

$$||P\delta_{(a,\lambda)}||^2=\overline{c}_1+O(\frac{1}{\lambda^{n-4}}),\quad \left(P\delta_{(a,\lambda)},\frac{\partial}{\partial\lambda}P\delta_{(a,\lambda)}\right)=O(\frac{1}{\lambda^{n-3}}),$$

$$\left(P\delta_{(a,\lambda)},\frac{\partial}{\partial a_i}P\delta_{(a,\lambda)}\right)=O(\frac{1}{\lambda^{n-4}}),\quad \left|\left|\frac{\partial}{\partial\lambda}P\delta_{(a,\lambda)}\right|\right|^2=\frac{\overline{c}_2}{\lambda^2}+O(\frac{1}{\lambda^{n-2}}),$$

$$\left(\frac{\partial}{\partial\lambda}P\delta_{(a,\lambda)},\frac{\partial}{\partial a_i}P\delta_{(a,\lambda)}\right)=O(\frac{1}{\lambda^{n-3}}),\quad \left(\frac{\partial}{\partial a_j}P\delta_{(a,\lambda)},\frac{\partial}{\partial a_i}P\delta_{(a,\lambda)}\right)=\overline{c}_3\lambda^2\delta_{ij}+O(\frac{1}{\lambda^{n-5}})$$

where $\overline{c}_i$'s are positive constants and δ_{ij} is the Kronecker symbol.
The other hand-side is given by $(\nabla J_\varepsilon(\alpha P\delta_{(a,\lambda)}+\overline{v}),\psi)$ where $\psi=P\delta_{(a,\lambda)}$, $(\partial P\delta_{(a,\lambda)})/(\partial\lambda)$, $(\partial P\delta_{(a,\lambda)})/(\partial a_i)$ with $i=1,...,n$.
Observe that, for $u=\alpha P\delta_{(a,\lambda)}+\overline{v}$,

$$(\nabla J_\varepsilon(u),\psi)=2J_\varepsilon(u)\left(\alpha(P\delta_{(a,\lambda)},\psi)-J_\varepsilon(u)^{(p+1-\varepsilon)/2}\int_\Omega u^{p-\varepsilon}\psi\right).$$

Expanding J_ε, we obtain

$$J_\varepsilon(\alpha P\delta_{(a,\lambda)}+\overline{v})=S+O\left(\varepsilon+\varepsilon Log\lambda+\frac{1}{\lambda^{n-4}}\right).\tag{3.2}$$

Now, using (2.10), (2.12), (2.16), (2.18), (2.20), (2.21), (2.22) and Proposition 2.6, we derive that, after taking the following change of variable: $\alpha=\alpha_0+\beta$,

$$\begin{aligned}(\nabla J_\varepsilon(u),P\delta)&=O\left(\varepsilon Log\lambda+|\beta|+\frac{1}{\lambda^{n-4}}\right)\\(\nabla J_\varepsilon(u),\partial P\delta/\partial\lambda)&=O\left(\frac{\varepsilon}{\lambda}+\frac{1}{\lambda^{n-3}}\right)\\(\nabla J_\varepsilon(u),\partial P\delta/\partial a_j)&=O\left(\lambda\varepsilon^2+\frac{1}{\lambda^{n-4}}\right),\ \text{for each } j=1,...,n.\end{aligned}$$

The solution of the system in A, B, and C_i's shows that

$$A=O\left(\varepsilon Log\lambda+|\beta|+\frac{1}{\lambda^{n-4}}\right),\quad B=O\left(\lambda\varepsilon+\frac{1}{\lambda^{n-5}}\right),\quad C_j=O\left(\frac{\varepsilon^2}{\lambda}+\frac{1}{\lambda^{n-2}}\right).$$

Now, to find critical points of K_ε, we have to solve the following system

$$(E_1)\qquad\begin{cases}\frac{\partial K_\varepsilon}{\partial\alpha}+\left(\frac{\partial K}{\partial v},\frac{\partial\bar{v}}{\partial\alpha}\right)&=0\\\frac{\partial K_\varepsilon}{\partial\lambda}+\left(\frac{\partial K}{\partial v},\frac{\partial\bar{v}}{\partial\lambda}\right)&=0\\\frac{\partial K_\varepsilon}{\partial a_j}+\left(\frac{\partial K}{\partial v},\frac{\partial\bar{v}}{\partial a_j}\right)&=0,\ \text{for } j=1,...,n.\end{cases}$$

Taking the derivatives, with respect to the different parameters on M_ε, of the following equalities

$$(V_0)\quad(\bar{v},P\delta_{a_\varepsilon,\lambda_\varepsilon})=(\bar{v},\partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial\lambda_i)=0,\ (\bar{v},\partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial a_i)=0 \text{ for } i=1,...,n$$

and using (3.1), we see that system (E_1) is equivalent to

$$(E_2) \qquad \begin{cases} \frac{\partial K_\varepsilon}{\partial \alpha} &= 0 \\ \frac{\partial K_\varepsilon}{\partial \lambda} &= B\left(\frac{\partial^2 P\delta}{\partial \lambda^2}, \overline{v}\right) + \sum_{i=1}^n C_i\left(\frac{\partial^2 P\delta}{\partial\lambda\partial a_i}, \overline{v}\right) \\ \frac{\partial K_\varepsilon}{\partial a_j} &= B\left(\frac{\partial^2 P\delta}{\partial\lambda\partial a_j}, \overline{v}\right) + \sum_{i=1}^n C_i\left(\frac{\partial^2 P\delta}{\partial a_i\partial a_j}, \overline{v}\right), \text{ for each } j = 1, ..., n. \end{cases}$$

The same computation as in the proof of Proposition 2.6 shows that

$$\begin{aligned} \frac{\partial K_\varepsilon}{\partial \alpha} &= (\nabla J_\varepsilon(\alpha P\delta + \overline{v}), P\delta) \\ &= 2J_\varepsilon(u)\left(\alpha S^{n/4}\left(1 - \alpha^{p-1}S^{n/(n-4)}\right) + O\left(\varepsilon Log\lambda + \frac{1}{\lambda^{n-4}}\right)\right). \end{aligned}$$

Furthermore, using the estimates provided in the proof of claim (a) of Proposition 2.7, we derive that

$$\begin{aligned} \lambda\frac{\partial K_\varepsilon}{\partial \lambda} &= \left(\nabla J_\varepsilon(\alpha P\delta + \overline{v}), \lambda\frac{\partial P\delta}{\partial\lambda}\right) \\ &= (n-4)J_\varepsilon(u)\left(\alpha c_1\frac{H(a,a)}{\lambda^{n-4}}\left(1 - 2\alpha^{p-1}S^{n/(n-4)}\right) + c_2 S^{\frac{n}{n-4}}\alpha^p\varepsilon\right. \\ &\left. + O\left(\varepsilon^2 Log\lambda + \frac{\varepsilon Log\lambda}{\lambda^{n-4}} + \frac{1}{\lambda^{n-2}}\right)\right). \end{aligned}$$

Following also the proof of claim (b) of Proposition 2.7, we obtain, for each $j = 1, ..., n$,

$$\begin{aligned} \frac{\partial K_\varepsilon}{\partial a_j} &= \left(\nabla J_\varepsilon(\alpha P\delta + \overline{v}), \frac{\partial P\delta}{\partial a_j}\right) \\ &= -\frac{c\alpha}{\lambda^{n-4}}\frac{\partial H}{\partial a}(a,a)\left(1 - 2\alpha^{p-1}S^{n/(n-4)}\right) + O\left(\lambda\varepsilon^2 + \frac{\varepsilon Log\lambda}{\lambda^{n-4}} + \frac{1}{\lambda^{n-2}} + (\text{if } n = 6)\frac{1}{\lambda^3}\right). \end{aligned}$$

On the other hand, one can easy verify that

$$(i)\ \|\frac{\partial^2 P\delta}{\partial\lambda^2}\| = O\left(\frac{1}{\lambda^2}\right), \qquad (ii)\ \|\frac{\partial^2 P\delta}{\partial\lambda\partial a_i}\| = O(1), \qquad (iii)\ \|\frac{\partial^2 P\delta}{\partial a_i\partial a_j}\| = O(\lambda^2). \tag{3.3}$$

Now, we take the following change of variables:

$$\alpha = \alpha_0 + \beta, \quad a = x_0 + \xi, \quad \frac{1}{\lambda^{\frac{n-4}{2}}} = \sqrt{\frac{c_2}{c_1}}\left(\frac{1}{H(x_0, x_0)} + \rho\right)\sqrt{\varepsilon}.$$

Then, using estimates (3.3), Proposition 2.6 and the fact that x_0 is a nondegenerate critical point of φ, the system (E_2) becomes

$$(E_3) \qquad \begin{cases} \beta &= O\left(\varepsilon|Log\varepsilon| + |\beta|^2\right) \\ \rho &= O\left(\varepsilon^{2/(n-4)} + |\beta|^2 + |\xi|^2 + \rho^2\right) \\ \xi &= O\left(\varepsilon^{2/(n-4)} + |\beta|^2 + |\xi|^2 + \rho^2 + (\text{if } n = 6)\varepsilon^{1/2}\right). \end{cases}$$

Thus Brower's fixed point theorem shows that the system (E_3) has a solution $(\beta^\varepsilon, \rho^\varepsilon, \xi^\varepsilon)$ for ε small enough such that

$$\beta^\varepsilon = O(\varepsilon|Log\varepsilon|), \quad \rho^\varepsilon = O(\varepsilon^{2/(n-4)} + (\text{if } n=6)\varepsilon^{1/2}), \quad \xi^\varepsilon = O(\varepsilon^{2/(n-4)} + (\text{if } n=6)\varepsilon^{1/2}).$$

By construction, the corresponding u_ε is a critical point of J_ε that is $w_\varepsilon = J_\varepsilon(u_\varepsilon)^{n/8}u_\varepsilon$ satisfies

$$\Delta^2 w_\varepsilon = |w_\varepsilon|^{8/(n-4)-\varepsilon} w_\varepsilon \text{ in } \Omega, \quad w_\varepsilon = \Delta w_\varepsilon = 0 \text{ on } \partial\Omega. \tag{3.4}$$

with $|w_\varepsilon^-|_{L^{2n/(n-4)}(\Omega)}$ very small, where $w_\varepsilon^- = \max(0, -w_\varepsilon)$.
As in Propostion 4.1 of [9], we prove that $w_\varepsilon^- = 0$. Thus, since w_ε is a non-negative function which satisfies (3.4), the strong maximum principle ensures that $w_\varepsilon > 0$ on Ω and then u_ε is a solution of $(P_{-\varepsilon})$, which blows-up at x_0 as ε goes to zero. This ends the proof of Theorem 1.3.

4 Proof of Theorem 1.4

First of all, we can easily show that for u_ε satisfying the assumption of the theorem, there is a unique way to choose a_ε, λ_ε and v_ε such that

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon \tag{4.1}$$

with

$$\begin{cases} \alpha_\varepsilon \in \mathbb{R}, \quad \alpha_\varepsilon \to 1 \\ a_\varepsilon \in \Omega, \quad \lambda_\varepsilon \in \mathbb{R}_+^*, \quad \lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to +\infty \\ v_\varepsilon \to 0 \quad \text{in } E := H^2 \cap H_0^1(\Omega), \quad v_\varepsilon \in E_{a_\varepsilon,\lambda_\varepsilon} \end{cases} \tag{4.2}$$

and for any $(a,\lambda) \in \Omega \times \mathbb{R}_+^*$, $E_{(a,\lambda)}$ denotes the subspace of E defined by (2.7).
In the following, we always assume that u_ε, satisfying the assumption of Theorem 1.4, is written as in (4.1). To simplify the notations, we set $\delta_{a_\varepsilon,\lambda_\varepsilon} = \delta_\varepsilon$, $P\delta_{a_\varepsilon,\lambda_\varepsilon} = P\delta_\varepsilon$ and $\theta_{a_\varepsilon,\lambda_\varepsilon} = \theta_\varepsilon$.
Now we are going to estimate the v_ε occurring in (4.1).

Lemma 4.1 *Let u_ε satisfying the assumption of Theorem 1.4. Then we have*

$$(i) \int_\Omega |\Delta u_\varepsilon|^2 \to S^{n/4} \ ; \quad (ii) \int_\Omega u_\varepsilon^{p+1+\varepsilon} \to S^{n/4}$$

as $\varepsilon \to 0$, S denoting the Sobolev constant defined by (1.2).

Proof. We have

$$\begin{aligned} \int_\Omega |\Delta u_\varepsilon|^2 &= \int_\Omega |\Delta(\alpha_\varepsilon P\delta_\varepsilon + v_\varepsilon)|^2 \\ &= \alpha_\varepsilon^2 \int_\Omega |\Delta P\delta_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 \quad \text{since } v_\varepsilon \in E_{a_\varepsilon,\lambda_\varepsilon}. \end{aligned}$$

From the fact that δ_ε satisfies $\Delta^2\delta_\varepsilon = \delta_\varepsilon^p$ in $\mathbb{R}^n$ and is a minimizer for S, we deduce that

$$\int_{\mathbb{R}^n} |\Delta\delta_\varepsilon|^2 = S^{n/4}.$$

On the other hand, an explicit computation provides us with

$$\int_\Omega |\Delta\delta_{a,\lambda}|^2 = \int_{\mathbb{R}^n} |\Delta\delta_{a,\lambda}|^2 + O\left(\frac{1}{(\lambda d(a,\partial\Omega))^n}\right) \quad \text{as } \lambda d(a,\partial\Omega) \to +\infty.$$

Using Proposition 2.1, claim (i) is a consequence of (4.2). Claim (ii) follows from the fact that u_ε solves $(P_{+\varepsilon})$. □

Lemma 4.2 *Let u_ε satisfying the assumption of Theorem 1.4. Then λ_ε occuring in* (4.1) *satisfies*

$$\lambda_\varepsilon^\varepsilon \to 1 \quad \text{as } \varepsilon \to 0.$$

Proof. According to Lemma 4.1, we have

$$\int_\Omega u_\varepsilon^{p+1+\varepsilon} = S^{n/4} + o(1) \quad \text{as } \varepsilon \to 0 \tag{4.3}$$

and

$$\begin{aligned}\int_\Omega u_\varepsilon^{p+1+\varepsilon} &= \int_\Omega (\alpha_\varepsilon P\delta_\varepsilon + v_\varepsilon)^{p+\varepsilon}\alpha_\varepsilon P\delta_\varepsilon + \int_\Omega u_\varepsilon^{p+\varepsilon} v_\varepsilon \\ &= \alpha_\varepsilon^{p+\varepsilon+1}\int_\Omega P\delta_\varepsilon^{p+\varepsilon+1} + \int_\Omega \Delta^2 u_\varepsilon v_\varepsilon + O\left(\int_\Omega P\delta_\varepsilon^{p+\varepsilon}|v_\varepsilon| + \int_\Omega |\, v_\varepsilon \,|^{p+\varepsilon} P\delta_\varepsilon\right) \\ &= \alpha_\varepsilon^{p+\varepsilon+1}\int_\Omega P\delta_\varepsilon^{p+\varepsilon+1} + O\left(\lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}\int_\Omega P\delta_\varepsilon^{p}|v_\varepsilon| + \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}\int_\Omega |\, v_\varepsilon \,|^{p+\varepsilon} P\delta_\varepsilon^{1-\varepsilon} + ||v_\varepsilon||\right) \\ &= \alpha_\varepsilon^{p+\varepsilon+1}\int_\Omega P\delta_\varepsilon^{p+\varepsilon+1} + O\left(\lambda_\varepsilon^{\varepsilon(n-4)/2}|v_\varepsilon|_{L^{p+1}} + \lambda_\varepsilon^{\varepsilon(n-4)/2}|v_\varepsilon|^{p+\varepsilon}_{L^{p+1}} + ||v_\varepsilon||\right).\end{aligned}$$

Thus

$$\int_\Omega u_\varepsilon^{p+1+\varepsilon} = \alpha_\varepsilon^{p+\varepsilon+1}\int_\Omega P\delta_\varepsilon^{p+\varepsilon+1} + o\left(\lambda_\varepsilon^{\varepsilon(n-4)/2} + 1\right). \tag{4.4}$$

We observe that

$$\begin{aligned}\int_\Omega P\delta_\varepsilon^{p+1+\varepsilon} &= \int_\Omega (\delta_\varepsilon - \theta_\varepsilon)^{p+1+\varepsilon} = \int_\Omega \delta_\varepsilon^{p+1+\varepsilon} + O\left(\int_\Omega \delta_\varepsilon^{p+\varepsilon}\theta_\varepsilon\right) \\ &= c_0^{p+\varepsilon+1}\int_{\mathbb{R}^n}\left(\frac{\lambda_\varepsilon}{1+\lambda_\varepsilon^2|x-a_\varepsilon|^2}\right)^{n+\varepsilon(n-4)/2} \\ &+ O\left(|\theta_\varepsilon|_{L^\infty}\int_\Omega\left(\frac{\lambda_\varepsilon}{1+\lambda_\varepsilon^2|x-a_\varepsilon|^2}\right)^{\frac{(p+\varepsilon)(n-4)}{2}} + \frac{\lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}}{(\lambda_\varepsilon d_\varepsilon)^n}\right).\end{aligned}$$

Using Proposition 2.1, we obtain

$$\int_\Omega P\delta_\varepsilon^{p+1+\varepsilon} = c_0^{p+1+\varepsilon}\lambda_\varepsilon^{\varepsilon(n-4)/2}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{n+\varepsilon(n-4)/2}} + O\left(\frac{\lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right).$$

We note that

$$c_0^{p+1+\varepsilon}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{n+\varepsilon(n-4)/2}} = c_0^{p+1}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^n}+O(\varepsilon)$$
$$= S^{n/4}+O(\varepsilon).$$

Therefore

$$\int_\Omega P\delta_\varepsilon^{p+1+\varepsilon} = \lambda_\varepsilon^{\varepsilon(n-4)/2}\Big(S^{n/4}+O(\varepsilon)+o(1)\Big). \tag{4.5}$$

and (4.4) and (4.5) provide us with

$$\int_\Omega u_\varepsilon^{p+1+\varepsilon} = \alpha_\varepsilon^{p+1+\varepsilon}\lambda_\varepsilon^{\varepsilon(n-4)/2}\Big(S^{n/4}+o(1)\Big)+o(1). \tag{4.6}$$

Combination of (4.3) and (4.6) proves the lemma. □

Next, as in Lemma 2.3 of [7], we can easily prove the following estimate :

Lemma 4.3 $\lambda_\varepsilon^\varepsilon = 1+o(1)$ *as* ε *goes to zero implies that*

$$\delta_\varepsilon^\varepsilon(x) - c_0^\varepsilon\lambda_\varepsilon^{\varepsilon(n-4)/2} = O\left(\varepsilon Log(1+\lambda_\varepsilon^2|x-a_\varepsilon|^2)\right) \qquad \text{in } \Omega.$$

We are now able to study the v_ε-part of u_ε.

Lemma 4.4 *Let* u_ε *satisfying the assumption of Theorem 1.4. Then* v_ε *occuring in* (4.1) *satisfies*

$$\int_\Omega |v_\varepsilon|^{p+1+\varepsilon} = o(1) \quad \text{as } \varepsilon\to 0.$$

Proof. We observe that

$$\int_\Omega u_\varepsilon^{p+1+\varepsilon} = \int_\Omega(\alpha_\varepsilon P\delta_\varepsilon)^{p+1+\varepsilon}+\int_\Omega|v_\varepsilon|^{p+1+\varepsilon}+O\left(\int_\Omega(\alpha_\varepsilon P\delta_\varepsilon)^{p+\varepsilon}|v_\varepsilon|+\int_\Omega|v_\varepsilon|^{p+\varepsilon}\alpha_\varepsilon P\delta_\varepsilon\right).$$

We are going to estimate each term of the right hand-side in the above equality.

$$\int_\Omega(\alpha_\varepsilon P\delta_\varepsilon)^{p+1+\varepsilon} = \alpha_\varepsilon^{p+1+\varepsilon}\left[\int_\Omega\delta_\varepsilon^{p+1+\varepsilon}+O\left(\int_\Omega\delta_\varepsilon^{p+\varepsilon}\theta_\varepsilon\right)\right]$$
$$= \alpha_\varepsilon^{p+1+\varepsilon}\int_\Omega\delta_\varepsilon^{p+1+\varepsilon}+o\left(\lambda_\varepsilon^{\varepsilon(n-4)/2}\right)$$
$$\int_\Omega(\alpha_\varepsilon P\delta_\varepsilon)^{p+\varepsilon}|v_\varepsilon| \le \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}||v_\varepsilon|| = o(1)$$
$$\int_\Omega|v_\varepsilon|^{p+\varepsilon}\alpha_\varepsilon P\delta_\varepsilon \le \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}}||v_\varepsilon||^{p+\varepsilon} = o(1)$$

using Proposition 2.1, Holder inequality and Sobolev embedding theorem. From Lemma 4.1 we derive that

$$S^{n/4} + o(1) = (1 + o(1)) \int_\Omega \delta_\varepsilon^{p+1+\varepsilon} + \int_\Omega |v_\varepsilon|^{p+1+\varepsilon}. \tag{4.7}$$

As we have also

$$\int_\Omega \delta_\varepsilon^{p+1+\varepsilon} = \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}} c_0^\varepsilon \int_\Omega \delta_\varepsilon^{p+1} + \int_\Omega \left(\delta_\varepsilon^{p+1+\varepsilon} - c_0^\varepsilon \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}} \delta_\varepsilon^{p+1} \right),$$

from Lemma 4.3 we deduce that

$$\begin{aligned} \int_\Omega \delta_\varepsilon^{p+1+\varepsilon} &= \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}} c_0^\varepsilon \left(S^{n/4} - \int_{\mathbb{R}^n \setminus \Omega} \delta_\varepsilon^{p+1} \right) + O\left(\varepsilon \int_\Omega \delta_\varepsilon^{p+1} Log(1 + \lambda_\varepsilon^2 |x - a_\varepsilon|^2) \right) \\ &= (1 + o(1)) S^{n/4} + O\left((\lambda_\varepsilon d_\varepsilon)^{-n} \right) + O(\varepsilon). \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain the desired result. □

Lemma 4.5 *Let u_ε satisfying the assumption of Theorem 1.4. Then we have*

$$\textbf{(i)} \quad |u_\varepsilon|_{L^\infty(\Omega)}^\varepsilon = O(1) \qquad \textbf{(ii)} \quad |v_\varepsilon|_{L^\infty(\Omega)}^\varepsilon = O(1),$$

where v_ε is defined in (4.1)*.*

Proof. We notice that Claim (ii) follows from Claim (i) and Lemma 4.2. Then we only need to show that Claim (i) is true. We define the rescaled functions

$$\omega_\varepsilon(y) = M_\varepsilon^{-1} u_\varepsilon \left(x_\varepsilon + M_\varepsilon^{(1-p-\varepsilon)/4} y \right), \quad y \in \Omega_\varepsilon = M_\varepsilon^{\frac{p-1+\varepsilon}{4}} (\Omega - x_\varepsilon), \tag{4.9}$$

where $x_\varepsilon \in \Omega$ is such that

$$M_\varepsilon := u_\varepsilon(x_\varepsilon) = |u_\varepsilon|_{L^\infty(\Omega)}. \tag{4.10}$$

ω_ε satisfies

$$\begin{cases} \Delta^2 \omega_\varepsilon = \omega_\varepsilon^{p+\varepsilon}, & 0 < \omega_\varepsilon \le 1 \quad \text{in} \quad \Omega_\varepsilon \\ \omega_\varepsilon(0) = 1, & \Delta\omega = \omega_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{4.11}$$

Following the same argument as in Lemma 2.3 [8], we have

$$M_\varepsilon^{(p-1+\varepsilon)/4} d(x_\varepsilon, \partial\Omega) \to +\infty \quad \text{as } \varepsilon \to 0.$$

Then it follows from standard elliptic theory that there exists a positive function ω such that (after passing to a subsequence) $\omega_\varepsilon \to \omega$ in $C^4_{loc}(\mathbb{R}^n)$, and ω satisfies

$$\begin{cases} \Delta^2 \omega = \omega^p, & 0 \le \omega \le 1 \quad \text{in } \mathbb{R}^n \\ \omega(0) = 1, & \nabla\omega(0) = 0. \end{cases}$$

It follows from [19] that ω writes as

$$\omega(y) = \delta_{0,\alpha_n}(y), \quad \text{with } \alpha_n = c_0^{2/(4-n)}$$

Therefore

$$M_\varepsilon^{\frac{\varepsilon(n-4)}{4}} \int_{B(x_\varepsilon, M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} u_\varepsilon^{p+1+\varepsilon}(x)dx = \int_{B(0,1)} \omega_\varepsilon^{p+1+\varepsilon}(y)dy \to c > 0 \quad \text{as } \varepsilon \to 0. \tag{4.12}$$

We notice that, as in the proof of Lemma 4.2

$$\begin{aligned}\int_{B(x_\varepsilon, M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} u_\varepsilon^{p+1+\varepsilon}(x)dx &= \int_{B(x_\varepsilon, M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} (\alpha_\varepsilon P\delta_\varepsilon + v_\varepsilon)^{p+1+\varepsilon}(x)dx \\ &= \alpha_\varepsilon^{p+1+\varepsilon} \int_{B(x_\varepsilon, M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} \delta_{a_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon}(x)dx + o(1) \\ &= \alpha_\varepsilon^{p+1+\varepsilon} \lambda_\varepsilon^{\frac{\varepsilon(n-4)}{2}} \int_{B(x_\varepsilon, \frac{\lambda_\varepsilon}{M_\varepsilon^{\frac{p-1+\varepsilon}{4}}})} \frac{dy}{(1+|y-\lambda_\varepsilon(a_\varepsilon - x_\varepsilon)|^2)^{n+\frac{\varepsilon(n-4)}{2}}} + o(1). \end{aligned} \tag{4.13}$$

Combining (4.12), (4.13) and Lemma 4.2, we obtain

$$M_\varepsilon^{\frac{\varepsilon(n-4)}{4}} \int_{B(0,\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} \frac{dy}{(1+|y-\lambda_\varepsilon(a_\varepsilon - x_\varepsilon)|^2)^{n+\frac{\varepsilon(n-4)}{2}}} \to c > 0 \quad \text{as } \varepsilon \to 0. \tag{4.14}$$

We have also

$$M_\varepsilon^{\frac{\varepsilon n}{4}} \int_{B(x_\varepsilon, M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} u_\varepsilon^{p+1}(x)dx = \int_{B(0,1)} \omega_\varepsilon^{p+1}(y)dy \to c > 0 \quad \text{as } \varepsilon \to 0.$$

Consequently, we find in the same way as above

$$M_\varepsilon^{\frac{\varepsilon n}{4}} \int_{B(0,\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}})} \frac{dy}{(1+|y-\lambda_\varepsilon(a_\varepsilon - x_\varepsilon)|^2)^{n}} \to c > 0 \quad \text{as } \varepsilon \to 0. \tag{4.15}$$

One of the two following cases occurs :

Case 1. $\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}} \not\to 0$ as $\varepsilon \to 0$. In this case, we can assume

$$\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}} \geq c_1 > 0, \quad \text{as } \varepsilon \to 0. \tag{4.16}$$

The claim follows from (4.16) and Lemma 4.2.

Case 2. $\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}} \to 0$ as $\varepsilon \to 0$. Now we distinguish two subcases:

Case 2.1. $\lambda_\varepsilon|a_\varepsilon - x_\varepsilon| \not\to +\infty$, as $\varepsilon \to 0$. We can assume that $\lambda_\varepsilon|a_\varepsilon - x_\varepsilon|$ remains bounded when $\varepsilon \to 0$. Thus, using (4.15) we obtain

$$M_\varepsilon^{\frac{\varepsilon n}{4}} \left(\lambda_\varepsilon M_\varepsilon^{\frac{1-p-\varepsilon}{4}}\right)^n \to c' > 0 \quad \text{as } \varepsilon \to 0,$$

which implies

$$\lambda_\varepsilon M_\varepsilon^{2/(n-4)} \to c'' > 0 \quad \text{as } \varepsilon \to 0.$$

Using Lemma 4.2, we derive a contradiction : this subcase cannot happen.

Case 2.2. $\lambda_\varepsilon |a_\varepsilon - x_\varepsilon| \to +\infty$ as $\varepsilon \to 0$. Using (4.14) and (4.15), we obtain

$$\frac{\lambda_\varepsilon^n}{\left(\lambda_\varepsilon |a_\varepsilon - x_\varepsilon|\right)^{2n+\varepsilon(n-4)} M_\varepsilon^{\varepsilon+\frac{2n}{n-4}}} \to C > 0 \quad \text{as } \varepsilon \to 0 \tag{4.17}$$

and

$$\frac{\lambda_\varepsilon^n}{\left(\lambda_\varepsilon |a_\varepsilon - x_\varepsilon|\right)^{2n} M_\varepsilon^{\frac{2n}{n-4}}} \to C > 0 \quad \text{as } \varepsilon \to 0. \tag{4.18}$$

From (4.17) and (4.18), we deduce that

$$M_\varepsilon^\varepsilon \left(\lambda_\varepsilon |a_\varepsilon - x_\varepsilon|\right)^{\varepsilon(n-4)} \to 1 \quad \text{as } \varepsilon \to 0 \tag{4.19}$$

and we also derive a contradiction. Consequently Case 2 cannot occur, and the lemma is proved. □

Now, arguing as in the proof of Proposition 2.6, we can easily derive the following estimate

Lemma 4.6 *Let u_ε satisfying the assumption of Theorem 1.4. Then v_ε occuring in* (4.1) *satisfies*

$$||v_\varepsilon|| \le C\left(\varepsilon + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}(\mathit{if}\ n < 12) + \frac{Log(\lambda_\varepsilon d_\varepsilon)}{(\lambda_\varepsilon d_\varepsilon)^4}(\mathit{if}\ n = 12) + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{\frac{n+4}{2}}}(\mathit{if}\ n > 12)\right)$$

with C independent of ε.

Now, using the same method in the proof of Claim (a) of Proposition 2.7, we can easily obtain the following result :

Proposition 4.7 *Let u_ε satisfying the assumption of Theorem 1.4. Then there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 \frac{H(a_\varepsilon, a_\varepsilon)}{\lambda_\varepsilon^{n-4}}(1 + o(1)) + C_2 \varepsilon (1 + o(1)) = O\left(\frac{Log(\lambda_\varepsilon d_\varepsilon)}{(\lambda_\varepsilon d_\varepsilon)^n} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^4}(\ \mathit{if}\ n = 5)\right)$$

where a_ε, λ_ε and $d_\varepsilon = d(a_\varepsilon, \partial\Omega)$ are given in (4.1).

We are now able to prove Theorem 1.4.

Proof of Theorem 1.4 Arguing by contradiction, let us suppose that $(P_{+\varepsilon})$ has a solution u_ε as stated in Theorem 1.4. From Proposition 4.7, we have

$$C_1 \frac{H(a_\varepsilon, a_\varepsilon)}{\lambda_\varepsilon^{n-4}}(1 + o(1)) + C_2 \varepsilon (1 + o(1)) = O\left(\frac{Log(\lambda_\varepsilon d_\varepsilon)}{(\lambda_\varepsilon d_\varepsilon)^n} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^4}(\text{ if } n = 5)\right) \tag{4.20}$$

with $C_1 > 0$ and $C_2 > 0$.

Two cases may occur :

Case 1. $d_\varepsilon \to 0$ as $\varepsilon \to 0$. Using (4.20) and the fact that $H(a_\varepsilon, a_\varepsilon) \sim c d_\varepsilon^{4-n}$, we derive a contradiction.

Case 2.$d_\varepsilon \not\to 0$ as $\varepsilon \to 0$. We have $H(a_\varepsilon, a_\varepsilon) \ge c > 0$ as $\varepsilon \to 0$ and (4.20) also leads to a contradiction. Thus our result follows. □

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